

# A STABLE RANGE DESCRIPTION OF THE SPACE OF LINK MAPS

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**ABSTRACT. Version: October 19, 2009** We study the space  $\text{Link}(P, Q; N)$  of link maps, maps from  $P \amalg Q$  to  $N$  such that the images of  $P$  and  $Q$  are disjoint. We give a range of dimensions, interpreted as the connectivity of a certain map, in which the cobordism class of the “linking manifold” is enough to distinguish the homotopy class of one link map from another.

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## 1. INTRODUCTION

Let  $N$  be a smooth manifold and let  $P$  and  $Q$  be smooth compact manifolds. A (smooth) *link map* of  $P$  and  $Q$  in  $N$  is a pair  $(f : P \rightarrow N, g : Q \rightarrow N)$  of smooth maps such that  $f(P)$  is disjoint from  $g(Q)$ . The set of link maps, denoted by  $\text{Link}(P, Q; N)$ , is an open subspace of  $\text{Map}(P, N) \times \text{Map}(Q, N) = \text{Map}(P \amalg Q, N)$ .

For brevity here we will write  $\mathcal{M}$  for  $\text{Map}(P, N) \times \text{Map}(Q, N)$  and denote the complement of the set of link maps in  $\mathcal{M}$  by  $\mathcal{B}$ . We prove that a certain “linking number” map

$$\ell : \text{hofiber}_{(f_1, g_1)}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M}) \rightarrow \Omega Q_+^{TN - (TP \oplus TQ)} \text{holim}(P \xrightarrow{f_1} N \xleftarrow{g_1} Q)$$

is  $(2(n - p - q) - 3)$ -connected, where  $p$ ,  $q$ , and  $n$  are the dimensions of the manifolds. The map was defined by the second author in [5], although the version we reference below is a homotopy-theoretic one given in [3]. Its domain is the homotopy fiber of the inclusion  $\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M}$  with respect to any point  $(f_1, g_1) \in \mathcal{M} - \mathcal{B}$ . Its codomain is the infinite loop space associated to the Thom spectrum of a virtual vector bundle. Both of these

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spaces are  $(n - p - q - 2)$ -connected. In the case when  $p + q = n - 1$  it was shown in [5] that the effect of the map  $\ell$  on  $\pi_0$  can be interpreted as a generalized linking number.

Remark: Functor calculus (the manifold version developed in [6, 2]) offers one point of view on link maps. Consider the functor  $(U, V) \mapsto \text{Link}(U, V; N)$  whose domain is the poset  $\mathcal{O}(P \amalg Q) = \mathcal{O}(P) \times \mathcal{O}(Q)$  of open subsets of  $P \amalg Q$ . Its best linear approximation is  $\text{Map}(U, N) \times \text{Map}(V, N)$ . Our result can be interpreted as a statement about a quadratic approximation to the same functor, but we will not pursue this here. This work overlaps the recent work of Klein and Williams [3]. In particular, some of the material in Section 3 also appears in [3].

Our main result is:

**Theorem 1.1.** *The map*

$$\Lambda : \Sigma \text{hofiber}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M}) \rightarrow Q_+^{TN-(TP \oplus TQ)} \text{holim}(P \rightarrow N \leftarrow Q)$$

*adjoint to  $\ell$  is  $(2(n - p - q) - 1)$ -connected.*

The fact that  $\ell$  is  $(2(n - p - q) - 3)$ -connected then follows immediately by the Freudenthal theorem, since the domain of  $\ell$  is  $(n - p - q - 1)$ -connected. Note that the connectivity claimed for  $\Lambda$  is negative if  $p + q \geq n$ , so it is no loss to assume  $p + q < n$ .

**1.1. Conventions.** A space  $X$  is  $k$ -connected if for every  $j$  with  $-1 \leq j \leq k$  every map  $S^j \rightarrow X$  can be extended to a map  $D^{j+1} \rightarrow X$ . (In other words,  $(-1)$ -connected means nonempty and if  $k \geq 0$  then  $k$ -connected means that there is exactly one path-component and that the homotopy groups vanish through dimension  $k$ .) A map is  $k$ -connected if each of its homotopy fibers is  $(k - 1)$ -connected. A (weak) *equivalence* is an  $\infty$ -connected map.

We write  $QX = \Omega^\infty \Sigma^\infty X$  if  $X$  is a based space. If  $X$  is unbased, then  $X_+$  means  $X$  with a disjoint basepoint added and  $Q_+X$  means  $Q(X_+)$ . For a vector bundle  $\xi$  over a space  $X$ , the unit disk bundle and the unit sphere bundle are  $D(X; \xi)$  and  $S(X; \xi)$ . The Thom space  $X^\xi$  is the quotient  $D(X; \xi)/S(X; \xi)$ , or equivalently the homotopy cofiber of the projection  $S(X; \xi) \rightarrow X$ . If  $\xi$  and  $\eta$  are two vector bundles on  $X$ , then by choosing a vector bundle monomorphism  $\eta \rightarrow \epsilon^i$  to a trivial bundle we can define  $Q_+^{\xi-\eta}X = \Omega^i QX^{\xi \oplus \epsilon^i/\eta}$ . This is essentially independent of the choice of  $i \geq 0$  and vector bundle monomorphism, in the sense that for large  $i$  the weak homotopy type of this space is independent of those choices.

## 2. SKETCH OF THE PROOF OF THEOREM 1.1

To prove Theorem 1.1 we will use the diagram (1) below and obtain the connectivity of  $\Lambda$  from the connectivities of all the other maps. For this we must introduce another closed set  $\mathcal{V} \subset \mathcal{B}$ . Recall that a point  $(f, g) \in \mathcal{M}$  belongs to  $\mathcal{B}$  if the statement  $f(x) = z = g(y)$  holds for some pair  $(x, y) \in P \times Q$  and some point  $z \in N$ . The closed set  $\mathcal{B}$  has codimension  $n - p - q$  in  $\mathcal{M}$  in some sense. Inside this space of “bad” maps is a set  $\mathcal{V}$  of “very bad” maps, having codimension in  $\mathcal{M}$  is  $2(n - p - q)$ . A point  $(f, g)$  is in  $\mathcal{V}$  if either the statement  $f(x) = z = g(y)$  holds for more than one choice of  $(x, y, z)$  or else it holds for one such choice in such a way that the associated map of tangent spaces  $T_x P \oplus T_y Q \rightarrow T_z N$  is not injective. The set  $\mathcal{B} - \mathcal{V}$  may be regarded as a submanifold of  $\mathcal{M}$ . It has maps to  $P$ ,

$Q$ , and  $N$  given by  $x$ ,  $y$ , and  $z$ . Pulling back tangent bundles via these maps, we obtain vector bundles on  $\mathcal{B} - \mathcal{V}$ , which we will denote simply by  $TP$ ,  $TQ$ , and  $TN$ . There is also a monomorphism  $TP \oplus TQ \rightarrow TN$ , and its cokernel  $TN/(TP \oplus TQ)$  may be thought of as the normal bundle of  $\mathcal{B} - \mathcal{V}$  in  $\mathcal{M}$ .

The next result immediately implies Theorem 1.1.

**Theorem 2.1.** *In the diagram below, the maps  $F$  and  $H$  are equivalences, the maps  $G$ ,  $C$ , and  $D$  are  $(2(n - p - q) - 1)$ -connected, and the map  $E$  is  $(3(n - p - q) - 2)$ -connected.*

$$\begin{array}{ccc}
 (1) & \Sigma \operatorname{hofiber}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M}) & \xrightarrow{\Lambda} Q_+^{TN-(TP \oplus TQ)} \operatorname{holim}(P \rightarrow N \leftarrow Q) \\
 & \uparrow G & \uparrow D \\
 & \Sigma \operatorname{hofiber}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M} - \mathcal{V}) & Q_+^{TN-(TP \oplus TQ)} \operatorname{hofiber}(\mathcal{B} - \mathcal{V} \rightarrow \mathcal{M}) \\
 & \uparrow F & \uparrow H \\
 & \operatorname{hofiber}(\mathcal{B} - \mathcal{V} \rightarrow \mathcal{M} - \mathcal{V})^{TN/(TP \oplus TQ)} & Q \operatorname{hofiber}(\mathcal{B} - \mathcal{V} \rightarrow \mathcal{M})^{TN/(TP \oplus TQ)} \\
 & & \uparrow C \\
 & & \operatorname{hofiber}(\mathcal{B} - \mathcal{V} \rightarrow \mathcal{M})^{TN/(TP \oplus TQ)}
 \end{array}$$

$\xrightarrow{E}$

We now briefly define the maps in the diagram and explain about their connectivities. Steps that are sketchy here will be filled in in the following sections. Let  $c = n - p - q$ .

The equivalence  $F$  is essentially an instance of the following general fact. If  $Y$  is a smooth submanifold of  $X$  and also a closed subset, then the suspension of the homotopy fiber of the inclusion  $X - Y \rightarrow X$  is equivalent to the Thom space, over the homotopy fiber of  $Y \rightarrow X$ , of the normal bundle of  $Y$  in  $X$ . This general fact will be proved, and adapted to the present function-space setting, in Section 4.

$G$  is an inclusion map. Since  $\mathcal{V}$  has codimension  $2c$  in  $\mathcal{M}$ , the inclusion  $\mathcal{M} - \mathcal{V} \rightarrow \mathcal{M}$  is  $(2c - 1)$ -connected. (This will be worked out in detail in Section 3) Therefore the map of homotopy fibers is  $(2c - 2)$ -connected and the map  $G$  of suspensions is  $(2c - 1)$ -connected.

$E$  is a map of Thom spaces. For a  $k$ -connected map  $Z \rightarrow W$  of spaces and a vector bundle  $\xi$  on  $W$  with fiber dimension  $d$ , the associated map  $Z^\xi \rightarrow W^\xi$  is  $(k + d)$ -connected. In our case  $d = c$  and  $k = 2c - 2$ ; the inclusion of  $\operatorname{hofiber}(\mathcal{B} - \mathcal{V} \rightarrow \mathcal{M} - \mathcal{V})$  into  $\operatorname{hofiber}(\mathcal{B} - \mathcal{V} \rightarrow \mathcal{M})$  is  $(2c - 2)$ -connected, again because the inclusion of  $\mathcal{M} - \mathcal{V}$  into  $\mathcal{M}$  is  $(2c - 1)$ -connected.

$C$  is the canonical map  $Z \rightarrow QZ$ , where the space  $Z$  is  $(c - 1)$ -connected, being the Thom space of a vector bundle of rank  $c$ . By the Freudenthal Theorem, the map is  $(2c - 1)$ -connected.

The equivalence  $H$  is simply a matter of rewriting the Thom spectrum of a virtual vector bundle  $\xi - \eta$  as the suspension spectrum of the Thom space of  $\xi/\eta$  when  $\eta$  is a subbundle of  $\xi$ .

The map  $D$  arises from a  $(c-1)$ -connected map from  $\text{hofiber}(\mathcal{B} - \mathcal{V} \rightarrow \mathcal{M})$  to  $\text{holim}(P \rightarrow N \leftarrow Q)$ . To explain further, we introduce the space  $\tilde{\mathcal{B}}$  of all  $((f, g), x, y, z) \in \mathcal{M} \times P \times Q \times N$  such that  $f(x) = z = g(y)$ . Projection to  $\mathcal{M}$  gives a map from  $\tilde{\mathcal{B}}$  onto  $\mathcal{B}$ . Let  $\tilde{\mathcal{V}} \subset \tilde{\mathcal{B}}$  be the preimage of  $\mathcal{V}$ . The projection  $\tilde{\mathcal{B}} - \tilde{\mathcal{V}} \rightarrow \mathcal{B} - \mathcal{V}$  is an isomorphism. The inclusion  $\tilde{\mathcal{B}} - \tilde{\mathcal{V}} \rightarrow \tilde{\mathcal{B}}$  is  $(c-1)$ -connected for reasons of codimension (again, the details are in section 3), and therefore the induced map  $\text{hofiber}(\mathcal{B} - \mathcal{V} \rightarrow \mathcal{M}) \rightarrow \text{hofiber}(\tilde{\mathcal{B}} \rightarrow \mathcal{M}) \simeq \text{holim}(P \rightarrow N \leftarrow Q)$  is also  $(c-1)$ -connected. There are vector bundles  $TP$ ,  $TQ$ , and  $TN$  on  $\tilde{\mathcal{B}}$  pulling back to their namesakes on  $\mathcal{B} - \mathcal{V}$ . (The monomorphism  $df \oplus dg : TP \oplus TQ \rightarrow TN$  is not available on the  $\text{holim}(P \rightarrow N \leftarrow Q)$  side, which is why we switched from Thom spaces to Thom spectra).

We end this section with a brief account of the commutativity of diagram (1). First we need to define the map  $\Lambda$ . As mentioned in Section 1,  $\Lambda$  is adjoint to a map  $\ell : \text{hofiber}_{(f_1, g_1)}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M})$ , which is a composite defined as follows. Let  $(f_t, g_t) \in \text{hofiber}_{(f_1, g_1)}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M})$ . The map  $\mathcal{M} \rightarrow \text{Map}(P \times Q, N \times N)$  given by  $(f, g) \mapsto f \times g$  induces a map  $\text{hofiber}_{(f_1, g_1)}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M}) \rightarrow \text{hofiber}_{f_1 \times g_1}(\text{Map}(P \times Q, N \times N - \Delta_N) \rightarrow \text{Map}(P \times Q, N \times N))$ . We can identify the latter homotopy fiber as a space of sections as follows.

Let  $E = \text{holim}(P \times Q \xrightarrow{f_1 \times g_1} N \times N \leftarrow N \times N - \Delta_N)$ . The projection map  $E \rightarrow P \times Q$  is a fibration with fiber over  $(p, q)$  the space  $\Phi_2(N) = \text{hofiber}_{(f_1(p), g_1(q))}(N \times N - \Delta_N \rightarrow N \times N)$ . Let  $\Gamma(P \times Q, E)$  its space of sections. This space of sections has a preferred basepoint given by  $(f_1, g_1)$ . It is equivalent to  $\text{hofiber}_{f_1 \times g_1}(\text{Map}(P \times Q, N \times N - \Delta_N) \rightarrow \text{Map}(P \times Q, N \times N))$  by inspection. Let  $Q_f S_f E \rightarrow P \times Q$  be the fibration whose fibers are  $Q S \Phi_2(N)$ , where  $S$  stands for the unreduced suspension. The canonical map  $\Gamma(P \times Q, E) \rightarrow \Omega \Gamma(P \times Q, Q_f S_f E)$  is easily shown to be  $(2n - p - q - 1)$ -connected, and there is an equivalence  $\Omega \Gamma(P \times Q, Q_f S_f E) \simeq \Omega Q_+^{TN - (TP \oplus TQ)} \text{holim}(P \times Q \xrightarrow{f_1 \times g_1} N \times N \leftarrow \Delta_N)$  which is the identity on the loop coordinate. Moreover, there is a homeomorphism  $\text{holim}(P \times Q \xrightarrow{f_1 \times g_1} N \times N \leftarrow \Delta_N) \cong \text{holim}(P \xrightarrow{f_1} N \xleftarrow{g_1} Q)$ . The composite map  $\text{hofiber}_{(f_1, g_1)}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M}) \rightarrow \Omega Q_+^{TN - (TP \oplus TQ)} \text{holim}(P \xrightarrow{f_1} N \xleftarrow{g_1} Q)$  is the map  $\ell$ , and  $\Lambda$  is its adjoint. See [3, Section 9].

Now let  $(f_t, g_t, v) \in \text{hofiber}(\mathcal{B} - \mathcal{V} \rightarrow \mathcal{M} - \mathcal{V})^{TN/TP \oplus TQ}$ . Here  $v$  is a vector of length  $0 \leq |v| \leq 1$ , and  $(f_t, g_t, v)$  is identified to a point when  $|v| = 1$ . After applying the maps  $E, C, H$ , and  $D$  in diagram (1), it is clear that  $(f_t, g_t, v)$  is sent to  $((x, \beta, y), v) \in Q_+^{TN - (TP \oplus TQ)} \text{holim}(P \rightarrow N \leftarrow Q)$ , where  $(x_0, y_0) \in P \times Q$  is the unique pair such that  $f_0(x_0) = g_0(y_0)$  and  $\beta : I \rightarrow N$  is the path defined by  $\beta(s) = f_{1-2s}(x_0)$  for  $0 \leq s \leq 1/2$  and  $\beta(s) = g_{2s-1}(y_0)$  for  $1/2 \leq s \leq 1$ .

Now we must apply  $F, G$ , and  $\Lambda$  to  $(f_t, g_t, v)$ . A careful examination of the material in Section 4 reveals that  $F$  sends  $(f_t, g_t, v)$  to the point  $s \wedge (\tilde{f}_t, \tilde{g}_t)$ , where  $s = 1 - |v|$  and  $(\tilde{f}_t, \tilde{g}_t) \in \text{hofiber}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M})$  is defined as follows. For  $s \leq t \leq 1$ ,  $(\tilde{f}_t, \tilde{g}_t) = (f_{(t-s)/(1-s)}, g_{(t-s)/(1-s)})$ . For  $0 \leq t \leq s$ ,  $(\tilde{f}_t, \tilde{g}_t)$  has the following properties:  $(\tilde{f}_t, \tilde{g}_t) \in \mathcal{M} - \mathcal{B}$  for  $t < s$ ,  $(\tilde{f}_s, \tilde{g}_s) = (f_0, g_0)$  has a unique pair  $(x_0, y_0) \in P \times Q$  such that  $f_0(x_0) = g_0(y_0) = z_0 \in N$  and such that  $f'_0(x_0) - g'_0(y_0) \in T_{z_0}N$ , when projected to

$T_{z_0}/T_{x_0}P \oplus T_{y_0}Q$ , is equal to  $v$  (here  $f'_0$  and  $g'_0$  are the derivatives with respect to  $t$ ). From this description of  $F$  and the description of  $\Lambda$  above, the diagram commutes.

### 3. CODIMENSION AND CONNECTIVITY

The proof outlined above uses that the pair  $(\mathcal{M}, \mathcal{M} - \mathcal{V})$  is  $(2n - 2p - 2q - 1)$ -connected and that the pair  $(\tilde{\mathcal{B}}, \tilde{\mathcal{B}} - \tilde{\mathcal{V}})$  is  $(n - p - q - 1)$ -connected. We now justify these statements more carefully.

For the first, it suffices if for every smooth manifold  $W$  of dimension  $j < 2n - 2p - 2q$ , for every map of pairs  $\phi : (W, \partial W) \rightarrow (\mathcal{M}, \mathcal{M} - \mathcal{V})$ , there is a homotopy of pairs to a map that is disjoint from  $\mathcal{V}$ .

Consider the adjoint map  $\Phi : W \times (P \amalg Q) \rightarrow N$ . By a preliminary homotopy we can assume that  $\Phi$  is smooth, and we can make the homotopy small enough in the  $C^0$  sense so that it corresponds to a homotopy of pairs. If we can show that the condition  $\phi^{-1}(\mathcal{V}) = \emptyset$  holds for a dense set of all such smooth maps  $\Phi$ , then another small homotopy will complete the job. For the density statement we will use the multijet transversality theorem of Mather [4, Proposition 3.3] (which appears in [1] as Theorem 4.13).

Recall the setup: Two smooth maps  $\Phi, \Psi : X \rightarrow Y$  have the same  $m$ -jet at  $x \in X$  if  $\Phi(x) = \Psi(x)$  and  $\Phi$  and  $\Psi$  have the same derivatives through order  $m$ . Let  $X^{(r)} \subset X^r$  be the space of configurations of  $r$  distinct points in  $X$ . The maps  $\Phi$  and  $\Psi$  have the same  $m$ -multijet at  $(x_1, \dots, x_r) \in X^{(r)}$  if for every  $i \in \{1, \dots, r\}$  they have the same  $m$ -jet at  $x_i$ . The manifold  $J_m^{(r)}(X, Y)$  of multijets has a point for each  $r$ -tuple  $(x_1, \dots, x_r)$  and each equivalence class of maps as above. A smooth map  $\Phi : X \rightarrow Y$  determines a smooth map  $j_m^{(r)}(\Phi) : X^{(r)} \rightarrow J_m^{(r)}(X, Y)$ . The multijet transversality theorem asserts that, for every submanifold  $Z$  of  $J_m^{(r)}(X, Y)$ , the set of all  $\Phi$  such that  $j_m^{(r)}(\Phi)$  is transverse to  $Z$  is a countable intersection of dense open sets in the function space  $\text{Map}(X, Y)$ . It follows that such a set, or even the intersection of countably many such sets, is dense.

We now introduce various submanifolds  $Z$  of  $J_m^{(r)}(W \times (P \amalg Q), N)$ , for various values of  $r$  and  $m$ . The point is that the condition  $\phi^{-1}(\mathcal{V}) = \emptyset$  will hold if and only if for each of these the set  $j_m^{(r)}(\Phi)$  is disjoint from  $Z$ . The codimension of  $Z$  will always be big enough so that in order for  $j_m^{(r)}(\Phi)$  to be transverse to  $Z$  it must be disjoint. Therefore the theorem will guarantee that there are maps  $W \rightarrow \mathcal{M} - \mathcal{V}$  arbitrarily close to a given map  $\Phi : W \rightarrow \mathcal{M}$ .

Let  $k$  be the dimension of  $W$ . We consider the various ways in which  $\phi$  could hit  $\mathcal{V}$ .

- (1) There might exist distinct  $x_1$  and  $x_2$  in  $P$  and distinct  $y_1$  and  $y_2$  in  $Q$  such that for some  $w \in W$  we have  $\Phi(w, x_1) = \Phi(w, y_1)$  and  $\Phi(w, x_2) = \Phi(w, y_2)$ . Then the point

$$((w, x_1), (w, x_2), (w, y_1), (w, y_2)) \in (W \times (P \amalg Q))^{(4)}$$

maps into a certain submanifold of  $J_0^{(4)}(W \times (P \amalg Q), N)$  whose codimension is  $3k + 2n$ . (That is  $3j$  to make four points of  $W$  equal to each other and  $2n$  for two coincidences in  $N$ .) This codimension is greater than the dimension  $4k + 2p + 2q$  of

- (the relevant open and closed part of)  $(W \times (P \amalg Q))^{(4)}$ , so that transverse means disjoint.
- (2) There might exist distinct  $x_1$  and  $x_2$  in  $P$  and  $y$  in  $Q$  such that  $\Phi(w, x_1) = \Phi(w, y) = \Phi(w, x_2)$ . This leads to a submanifold of  $J_0^{(3)}(W \times (P \amalg Q), N)$  whose codimension is  $2k + 2n$ , greater than the dimension  $3k + 2p + q$  of (part of)  $(W \times (P \amalg Q))^{(3)}$ .
  - (3) There might exist  $x$  in  $P$  and distinct  $y_1$  and  $y_2$  in  $Q$  such that  $\Phi(w, x) = \Phi(w, y_1) = \Phi(w, y_2)$ . The relevant manifold has codimension  $2k + 2n$  in  $J_0^{(3)}(W \times (P \amalg Q), N)$ , greater than  $3k + p + 2q$ .
  - (4) There might exist  $x \in P$  and  $y \in Q$  such that  $\Phi(w, x) = z = \Phi(w, y)$  and such that the linear map  $T_x P \oplus T_y Q \rightarrow T_z N$  given by differentiation of  $\phi(w)$  at  $x$  and  $y$  has rank less than  $p + q$ . For each fixed rank  $r < p + q$  this leads to a submanifold of  $J_1^{(2)}(W \times (P \amalg Q), N)$  whose codimension  $k + (n - r)(p + q - r)$  is greater than  $2k + p + q$ .

This completes the proof that the pair  $(\mathcal{M}, \mathcal{M} - \mathcal{V})$  is  $(2n - 2p - 2q - 1)$ -connected.

To prove that the pair  $(\tilde{\mathcal{B}}, \tilde{\mathcal{B}} - \tilde{\mathcal{V}})$  is  $(n - p - q - 1)$ -connected, essentially the same kind of standard dimension-counting will succeed, but a simple reference as before to the multijet transversality theorem will not suffice because  $\tilde{\mathcal{B}}$  is not simply the space of maps from one manifold to another.

First observe that both the projection  $\tilde{\mathcal{B}} \rightarrow P \times Q \times N$  and its restriction  $\tilde{\mathcal{B}} - \tilde{\mathcal{V}} \rightarrow P \times Q \times N$  are fibrations. It therefore suffices if, for a point  $(x_0, y_0, z_0) \in P \times Q \times N$ , the pair  $(\tilde{\mathcal{B}}_0, \tilde{\mathcal{B}}_0 - \tilde{\mathcal{V}}_0)$  of fibers is  $(n - p - q - 1)$ -connected. Here  $\tilde{\mathcal{B}}_0 \subset \mathcal{M}$  is the set of all  $\phi$  such that  $\phi(x_0) = z_0 = \phi(y_0)$ , and  $\tilde{\mathcal{V}}_0 \subset \tilde{\mathcal{B}}_0$  is the set of all  $\phi$  such that in addition at least one of the following is true:

- (1)  $\phi(x) = \phi(y)$  for some  $x \in P - x_0$  and some  $y \in Q - y_0$
- (2)  $\phi(x) = z_0$  for some  $x \in P - x_0$
- (3)  $\phi(y) = z_0$  for some  $y \in Q - y_0$
- (4) The linear map  $T_{x_0} P \oplus T_{y_0} Q \rightarrow T_{z_0} N$  has rank less than  $p + q$ .

To deal first with (4), note that  $\tilde{\mathcal{B}}_0$  is fibered over the space  $\mathcal{L}$  of all linear maps  $T_{x_0} P \oplus T_{y_0} Q \rightarrow T_{z_0} N$ . Let  $\mathcal{L}^{max} \subset \mathcal{L}$  be the open set of maps of rank  $p + q$  and let  $\tilde{\mathcal{B}}_0^{max} \subset \tilde{\mathcal{B}}_0$  be its preimage. The pair  $(\tilde{\mathcal{B}}_0, \tilde{\mathcal{B}}_0^{max})$  is  $(n - p - q - 2)$ -connected (one better than needed), because the pair  $(\mathcal{L}, \mathcal{L}^{max})$  is  $(n - p - q - 2)$ -connected, because the closed set  $\mathcal{L} - \mathcal{L}^{max}$  is the union of finitely many submanifolds having codimension at least  $n - p - q - 1$ .

It remains to show that the pair  $(\tilde{\mathcal{B}}_0^{max}, \tilde{\mathcal{B}} - \tilde{\mathcal{V}})$  is  $(n - p - q - 1)$ -connected. Both  $\tilde{\mathcal{B}}_0^{max}$  and  $\tilde{\mathcal{B}} - \tilde{\mathcal{V}}$  fiber over  $\mathcal{L}^{max}$ , so we can replace the two spaces by their fibers, say  $\tilde{\mathcal{B}}_L$  and  $\tilde{\mathcal{B}}_L - \tilde{\mathcal{V}}_L$  over a given  $L \in \mathcal{L}$ .

Now given a map  $\phi : W \rightarrow \tilde{\mathcal{B}}_L$ , we want to perturb it slightly so as to eliminate behaviors (1), (2) and (3). None of these can occur for  $x$  near  $x_0$  or  $y$  near  $y_0$  anyway, given the choice of  $L$ , so we look for perturbations that are fixed near  $x_0$  and  $y_0$ . In other words, we look for a small compactly supported change in the map  $\Phi : W \times ((P - x_0) \amalg (Q - y_0)) \rightarrow N$ . This goes as before: case (1) leads to a submanifold of  $J_0^{(2)}(W \times ((P - x_0) \amalg (Q - y_0)), N)$  with

codimension  $j + n$ , greater than  $2j + p + q$ ; case (2) leads to a submanifold of  $J_0^{(1)}(W \times ((P - x_0) \amalg (Q - y_0)), N)$  with codimension  $n$ , greater than  $j + p$ ; and case (3) leads to a submanifold of  $J_0^{(1)}(W \times ((P - x_0) \amalg (Q - y_0)), N)$  with codimension  $n$ , greater than  $j + q$ .

#### 4. NORMAL BUNDLES AND HOMOTOPY COFIBERS

Suppose that  $X$  is a smooth manifold, and that the closed subset  $Y \subset X$  is a smooth submanifold with normal bundle  $\nu$ .

Of course, the Thom space  $Y^\nu$  is equivalent to the homotopy cofiber of the inclusion map  $X - Y \rightarrow X$ . This follows from the fact that there is a homotopy pushout square

$$(2) \quad \begin{array}{ccc} S(Y; \nu) & \longrightarrow & D(Y; \nu) \\ \downarrow & & \downarrow \\ X - Y & \longrightarrow & X. \end{array}$$

The homotopy fibers over  $X$  of the four spaces above form another homotopy pushout square

$$\begin{array}{ccc} \text{hofiber}(S(Y; \nu) \rightarrow X) & \longrightarrow & \text{hofiber}(D(Y; \nu) \rightarrow X) \\ \downarrow & & \downarrow \\ \text{hofiber}(X - Y \rightarrow X) & \longrightarrow & \text{hofiber}(X \rightarrow X) \simeq *. \end{array}$$

Comparing homotopy cofibers of the rows in this square, we obtain an equivalence

$$\text{hofiber}(Y \rightarrow X)^\nu \rightarrow \Sigma \text{hofiber}(X - Y \rightarrow X)$$

Here we have written  $\nu$  for the pullback of  $\nu$  to  $\text{hofiber}(Y \rightarrow X)$ .

We need statements like those above in which the manifolds  $X$  and  $Y$  are replaced by the function spaces  $\mathcal{M} - \mathcal{V}$  and  $\mathcal{B} - \mathcal{V}$  and the role of the normal bundle is played by the vector bundle  $TN/(TP \oplus TQ)$  on  $\mathcal{B} - \mathcal{V}$ . The only little difficulty is that the square (2) depended on having a tubular neighborhood. We will write down a substitute for (2) that avoids this dependence.

Let  $P(Y, X)$  be the space of all smooth paths  $\gamma : [0, 1] \rightarrow X$  such that  $\gamma^{-1}(Y) = 0$  and  $\gamma'(0)$  is not tangent to  $Y$ . We have the homotopy-commutative square

$$(3) \quad \begin{array}{ccc} P(Y, X) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X - Y & \longrightarrow & X \end{array}$$

in which the top and left maps are evaluation at 0 and at 1 respectively.

There are equivalences

$$(4) \quad \text{hocofiber}(P(Y, X) \rightarrow Y) \rightarrow \text{hocofiber}(S(Y; \nu) \rightarrow Y) = Y^\nu$$



and

$$(5) \quad \text{hocofiber}(P(Y, X) \rightarrow Y) \rightarrow \text{hocofiber}(X - Y \rightarrow X).$$

The logic is as follows:

For (4) we use the map  $P(Y, X) \rightarrow S(Y; \nu)$  that sends  $\gamma$  to the projection of  $\gamma'(0)$  in the direction perpendicular to  $Y$ , normalized to have unit length. It is a map over  $Y$  between two spaces fibered over  $Y$ , and it is an equivalence because for each point in  $Y$  the map of fibers is an equivalence.

For (5) we need to see that the homotopy-commutative square (3) is a homotopy pushout, in the sense that the associated map from the homotopy colimit of

$$X - Y \leftarrow P(Y, X) \rightarrow Y$$

to  $X$  is an equivalence. After choosing a tubular neighborhood of  $Y$  in  $X$ , one can map  $S(Y; \nu)$  to  $P(Y, X)$  by using radial paths perpendicular to  $Y$ . This map is an equivalence because it is a one-sided inverse to an equivalence. It follows that in showing that the square is a homotopy pushout we may consider instead the square

$$\begin{array}{ccc} S(Y; \nu) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X - Y & \longrightarrow & X \end{array}$$

But this comes down to considering the same strictly commutative square (2) that we began with.

Note that although a tubular neighborhood was used in proving (5) to be an equivalence, the definitions of (4) and (5) did not use it. This is the point of introducing  $P(Y, X)$ .

Now for the function spaces: Again we will obtain equivalences

$$\text{hocofiber}(P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V}) \rightarrow \mathcal{M} - \mathcal{V}) \rightarrow (\mathcal{B} - \mathcal{V})^\nu$$

(where  $\nu$  now means the bundle  $TN/(TP \oplus TQ)$  on  $\mathcal{B} - \mathcal{V}$ ) and

$$\text{hocofiber}(P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V}) \rightarrow \mathcal{M} - \mathcal{V}) \rightarrow \text{hocofiber}(\mathcal{M} - \mathcal{B} \rightarrow \mathcal{M} - \mathcal{V})$$

We define the space  $P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V})$ . A point in it is a map  $\gamma : [0, 1] \rightarrow \mathcal{M}$  meeting the following conditions. Write  $\gamma(t) = (f_t, g_t)$ . The conditions are:

- (1)  $\gamma$  is smooth in the sense that the adjoint maps  $(t, x) \mapsto f_t(x)$  and  $(t, x) \mapsto g_t(x)$  from  $[0, 1] \times P$  and  $[0, 1] \times Q$  to  $N$  are smooth.
- (2) For every  $t > 0$ ,  $\gamma_t$  is in  $\mathcal{M} - \mathcal{B}$ , that is,  $f_t(P) \cap g_t(Q) = \emptyset$ .
- (3)  $\gamma_0 \in \mathcal{B} - \mathcal{V}$ , that is, (3a) there is exactly one point  $(x_0, z_0, y_0) \in P \times N \times Q$  such that  $f_0(x) = z_0 = g_0(y)$  and (3b)  $df_0 \oplus dg_0 : T_{x_0}P \oplus T_{y_0}Q \rightarrow T_{z_0}N$  is injective.
- (4)  $\gamma'(0)$  is not tangent to  $\mathcal{B} - \mathcal{V}$ , that is, the vector  $f'_0(x_0) - g'_0(y_0) \in T_{z_0}(N)$  does not belong to the subspace  $(d_{x_0}f)(T_{x_0}P) \oplus (d_{y_0}g)(T_{y_0}Q)$ . Here  $f'$  and  $g'$  are derivatives with respect to  $t$ .



Consider the homotopy-commutative square

$$\begin{array}{ccc} P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V}) & \longrightarrow & \mathcal{B} - \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{M} - \mathcal{B} & \longrightarrow & \mathcal{M} - \mathcal{V}, \end{array}$$

where the upper map and the left map take  $\gamma = (f, g)$  to  $(f_0, g_0)$  and  $(f_1, g_1)$  respectively. We argue much as in the finite-dimensional case.

First, there is an equivalence  $P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V}) \rightarrow S(\mathcal{B} - \mathcal{V}; \nu)$  that respects the projection to  $\mathcal{B} - \mathcal{V}$ , namely the map that takes  $\gamma = (f, g)$  to the unit vector in  $T_{z_0}N/(T_{x_0}P \oplus T_{y_0}Q)$  determined by the element  $f'_0(x_0) - g'_0(y_0)$  of  $T_{x_0}P \oplus T_{y_0}Q$ . It is an equivalence because it is a map between spaces fibered over  $\mathcal{B} - \mathcal{V}$  and it induces equivalences fiber by fiber.

Second, the square is a homotopy pushout. For this step, instead of trying to come up with a tubular neighborhood we reduce to the finite-dimensional case.

To show that the map from the homotopy colimit of

$$\mathcal{M} - \mathcal{B} \leftarrow P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V}) \rightarrow \mathcal{B} - \mathcal{V}$$

to  $\mathcal{M} - \mathcal{V}$  is surjective on homotopy groups, let  $X = S^k$  and take any map  $\phi : X \rightarrow \mathcal{M} - \mathcal{V}$ , with adjoint  $\Phi = (F, G)$ ,  $F : X \times P \rightarrow N$ ,  $G : X \times Q \rightarrow N$ . Deforming by a homotopy that stays within  $\mathcal{M} - \mathcal{V}$ , make  $\Phi$  “transverse to  $\mathcal{B} - \mathcal{V}$ ” in the sense that  $F$  and  $G$  together give a map  $X \times P \times Q \rightarrow N \times N$  which is transverse to the diagonal. The preimage of the diagonal in  $X \times P \times Q$  is a submanifold, and it is embedded in  $X$  by the projection. Call its image  $Y$ . The normal bundle of  $Y$  in  $X$  is the pullback of  $TN/(TP \oplus TQ)$  by  $\phi$ .

Now inverting the equivalence

$$\mathrm{hocolim}(X - Y \leftarrow P(Y, X) \rightarrow Y) \rightarrow X$$

and composing with the obvious map

$$\mathrm{hocolim}(X - Y \leftarrow P(Y, X) \rightarrow Y) \rightarrow \mathrm{hocolim}(\mathcal{M} - \mathcal{B} \leftarrow P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V}) \rightarrow \mathcal{M} - \mathcal{V})$$

we get

$$X \rightarrow \mathrm{hocolim}(\mathcal{M} - \mathcal{B} \leftarrow P(\mathcal{B} - \mathcal{V}, \mathcal{M} - \mathcal{V}) \rightarrow \mathcal{M} - \mathcal{V})$$

a lifting (up to homotopy) of  $\phi$ . Essentially the same argument serves to lift a homotopy and prove the injectivity.

Taking homotopy fibers over  $\mathcal{M} - \mathcal{V}$  all around, we obtain the needed equivalence F.

## 5. ACKNOWLEDGMENTS

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